Steepening of Alfven waves due to parallel compressibility

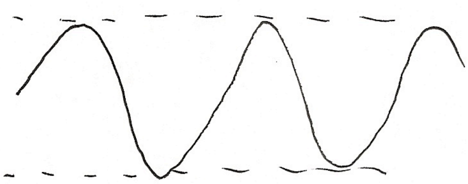
(Ref : "Relaxation Dynamics in Laboratory and Astrophysical Plasmas"
Ch 4.4 , P.H. Diamond , et. al. 2010)

Alfven waves $\Leftrightarrow$ $\nabla \cdot \underline{v} = 0$ $\not\longrightarrow$ Steepening

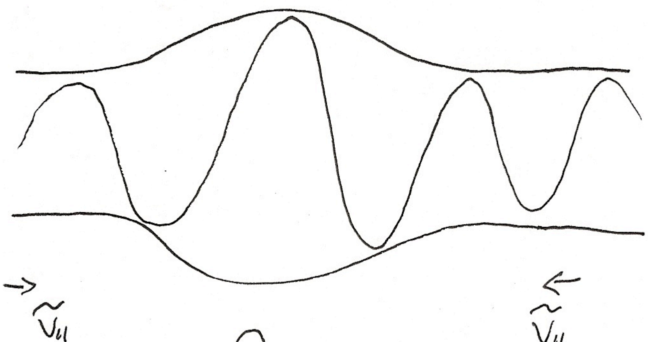
However, a little compressibility will lead to steepening of wave packet.

Approach : Modulational instability

Alfven wave :



Small compression :



$\Rightarrow$ Parallel flow $\tilde{v}_\parallel$

$\tilde{v}_\parallel \sim \tilde{B}^2$

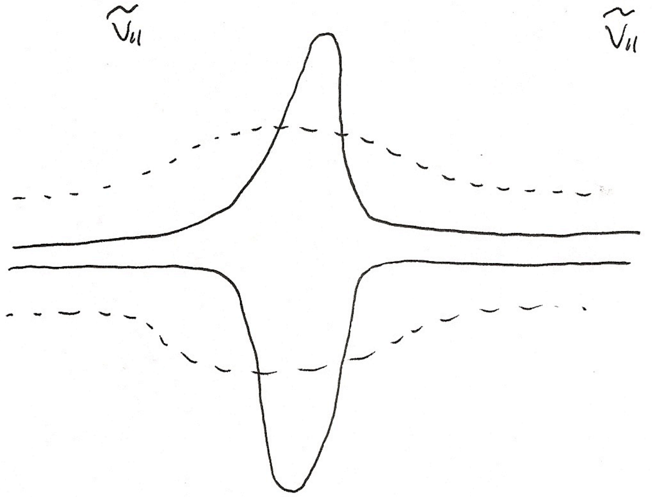
$\downarrow$

$\tilde{B}\uparrow$, $\frac{\partial}{\partial t}\tilde{B} = -\underline{v}\cdot\nabla\tilde{B}$

feedback loop

Result.

Steepening :



Until : Saturated by - dissipation η

- dispersion $d_i \sim \frac{c}{\omega_{pi}}$ ← ion inertial scale

Physical derivation :

Need equation for envelope $\Longleftrightarrow$ Scale separation $\nearrow$ fast $\searrow$ slow

Dispersion relation for Alfven wave with perturbed density $\rho = \rho_0 + \tilde{\rho}$:

$$\omega = k_{\parallel} V_A = k_{\parallel} \frac{B_0}{\sqrt{4\pi(\rho_0 + \tilde{\rho})}} \cong \underbrace{k_{\parallel} V_A}_{\text{fast}} - \underbrace{\frac{k_{\parallel} V_A}{2} \frac{\tilde{\rho}}{\rho_0}}_{\text{slow}} , \quad V_A = \frac{B_0}{\sqrt{4\pi\rho_0}}$$

$$\Rightarrow \quad \omega = \omega^{(0)} + i \left(\frac{\partial}{\partial t}\right)_{slow} , \quad k_{\parallel} = k_{\parallel}^{(0)} - i \left(\frac{\partial}{\partial x}\right)_{slow}$$

$$\Rightarrow \quad \frac{\partial}{\partial t} \delta B = - \frac{V_A}{2} \boxed{\frac{\tilde{\rho}}{\rho_0}} \left(\frac{\partial}{\partial x} \delta B\right) , \quad x, t \text{ are slow variables .}$$

Envelope eqn

Set by $\tilde{V}_{\parallel}$ dynamics

$$\frac{\partial}{\partial t} \tilde{\rho} = - \rho_0 \frac{\partial}{\partial x} \tilde{V}_{\parallel}$$

$$\rho_0 \frac{\partial}{\partial t} \tilde{V}_{\parallel} = - \frac{\partial}{\partial x} \tilde{P} - \frac{\partial}{\partial x} \left(\rho_0 \frac{\tilde{V}^2}{2} + \frac{\delta B^2}{8\pi} \right)$$

For Alfven wave, $\rho_0 \frac{\tilde{V}^2}{2} = \frac{\delta B^2}{8\pi}$

and $\tilde{P} = c_s^2 \tilde{\rho}$, $c_s^2 = \gamma \frac{P}{\rho}$

$$\Rightarrow \quad \frac{\partial^2}{\partial t^2} \tilde{V}_{\parallel} = c_s^2 \frac{\partial^2}{\partial x^2} \tilde{V}_{\parallel} - \frac{\partial^2}{\partial t \partial x} \frac{\delta B^2}{4\pi\rho_0}$$

Transfer to the frame of wave moving , $\tilde{V}_{\parallel} = \tilde{V}_{\parallel}(x - V_A t)$

$$\tilde{V}_{\parallel} = \frac{V_A}{V_A^2 - c_s^2} \frac{\delta B^2}{4\pi\rho_0} , \quad \frac{\tilde{\rho}}{\rho_0} = \frac{\tilde{V}_{\parallel}}{V_A} = \frac{1}{1-\beta} \frac{\delta B^2}{B_0^2} , \quad \beta = \frac{c_s^2}{V_A^2}$$

Plug $\frac{\tilde{\rho}}{\rho_0}$ into the envelope eqn :

$$\frac{\partial}{\partial t} \delta B = - \frac{V_A}{2} \frac{1}{1-\beta} \boxed{\frac{\delta B^2}{B_0^2}} \frac{\partial}{\partial x} \delta B + \boxed{???}$$

Steepening v.s. ?? (What stops steepening?)

Steepening is stopped by :

① Dissipation : $\eta \frac{\partial^2}{\partial x^2} \delta B$ ← small

② Dispersion : $i d_i^2 \Omega_i \frac{\partial^2}{\partial x^2} \delta B$ ← ion inertial scale physics.

↘ Quasi-parallel Alfvenic collisionless shock

Full envelope eqn :

$$\frac{\partial}{\partial t}\delta B = -\frac{v_A}{2}\frac{1}{1-\beta}\frac{\delta B^2}{B_0^2}\frac{\partial}{\partial x}\delta B + \eta\frac{\partial^2}{\partial x^2}\delta B + i d_i^2 \Omega_i \frac{\partial^2}{\partial x^2}\delta B$$

which is a derivative nonlinear Schrödinger (DNLS) equation.